

Modular Models for Intuitionistic Justification Logic

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Abstract

We present the logic iJT4, which is an explicit version of intuitionistic S4 and establish soundness and completeness with respect to modular models.

1 Introduction

Justification logics are explicit modal logics in the sense that they unfold the \Box -modality in families of so-called justification terms. Instead of formulas $\Box A$, meaning that A is known, justification logics include formulas $t : A$, meaning that A is known for reason t .

Artemov's original semantics for the first justification logic, the Logic of Proofs LP, was a provability semantics that interpreted $t : A$ roughly as *t represents a proof of A* in the sense of a formal proof predicate in Peano Arithmetic [1, 2, 17].

Later Fitting [12] interpreted justifications as evidence in a more general sense and introduced epistemic, i.e., possible world, models for justification logics. These models have been further developed to modular models as we use them in this paper [5, 15]. This general reading of justification led to many applications in epistemic logic [3, 4, 7, 8, 9, 10, 13, 14, 16].

Given the interpretation of LP in Peano Arithmetic, it was a natural question to find an intuitionistic version iLP of LP that is the logic of proofs of Heyting arithmetic. The work by Artemov and Iemhoff [6] and later by Dashkov [11] provides such an iLP. It turned out that iLP is not only LP with the underlying logic changed to intuitionistic propositional logic. In order to get a complete axiomatization with respect to provability semantics, one also has to include certain admissible rules of Heyting arithmetic as axioms in iLP so that they are represented by novel proof terms.

The main contribution of the present paper is that these additional axioms are not needed if we are interested in completeness with respect to modular models. We introduce the intuitionistic justification logic iJT4_{CS} , which is simply LP over an intuitionistic base instead of a classical one but without any additional axioms. We introduce possible world models for iJT4_{CS} that are inspired by the Kripke semantics for intuitionistic S4 and establish completeness of iJT4_{CS} with respect to these models.

2 Intuitionistic Justification Logic

In this section, we introduce the syntax for the justification logic iJT4_{CS} , which is the explicit analogue of the intuitionistic modal logic iS4 .

Definition 2.1 (Justification Terms). We assume a countable set of justification constants and a countable set of justification variables. Justification terms are inductively defined by:

1. each justification constant and each justification variable is a justification term;
2. if s and t are justification terms, then so are
 - $(s \cdot t)$, read s dot t ,
 - $(s + t)$, read s plus t ,
 - $!s$, read bang s .

We denote the set of terms by Tm .

Definition 2.2 (Formulas). We assume a countable set Prop of atomic propositions. The set of formulas \mathcal{L}_{J} is inductively defined by:

1. every atomic proposition is a formula;
2. the constant symbol \perp is a formula;
3. If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulas;
4. if A is a formula and t a term, then $t : A$ is a formula.

Definition 2.3. The axioms of iJT4 consist of the following axioms:

1. all axioms for intuitionistic propositional logic

2. $t : (A \rightarrow B) \rightarrow (s : A \rightarrow t \cdot s : B)$
3. $t : A \rightarrow t + s : A$ and $s : A \rightarrow t + s : A$
4. $t : A \rightarrow A$
5. $t : A \rightarrow !t : t : A$

A *constant specification* \mathbf{CS} is any subset

$$\mathbf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an axiom of iJT4}\}.$$

A constant specification \mathbf{CS} is called *axiomatically appropriate* if for each axiom A of iJT4, there is a constant c such that $(c, A) \in \mathbf{CS}$.

For a constant specification \mathbf{CS} the deductive system $\mathbf{iJT4}_{\mathbf{CS}}$ is the Hilbert system given by the axioms above and by the rules modus ponens and axiom necessitation:

$$\frac{A \quad A \rightarrow B}{B} \quad \frac{(c, A) \in \mathbf{CS}}{c : A}$$

As usual in justification logic, we can establish the deduction theorem and the internalization property.

Theorem 2.4 (Deduction Theorem). For every set of formulas M and all formulas A, B we have that

$$M \cup \{A\} \vdash_{\mathbf{iJT4}_{\mathbf{CS}}} B \iff M \vdash_{\mathbf{iJT4}_{\mathbf{CS}}} A \rightarrow B.$$

Lemma 2.5 (Internalization for Arbitrary Terms). Let \mathbf{CS} be an axiomatically appropriate constant specification. For arbitrary formulas A, B_1, \dots, B_n and arbitrary terms s_1, \dots, s_n , if

$$B_1, \dots, B_n \vdash_{\mathbf{iJT4}_{\mathbf{CS}}} A,$$

then there is a term $t \in \mathbf{Tm}$ such that

$$s_1 : B_1, \dots, s_n : B_n \vdash_{\mathbf{iJT4}_{\mathbf{CS}}} t : A.$$

3 Basic Modular Models

Basic modular models are syntactic models for justification logic. Yet, our basic modular models will include possible worlds in order to deal with the intuitionistic base logic. After defining basic modular models for intuitionistic justification logic, we will prove soundness and completeness.

Definition 3.1 (Basic evaluation). A *basic evaluation* is a tuple $(W, \leq, *)$ where

$W \neq \emptyset$ and \leq is a partial order on W ,

$$* : \mathbf{Prop} \times W \rightarrow \{0, 1\} \quad * : \mathbf{Tm} \times W \rightarrow \mathcal{P}(\mathcal{L}_J)$$

(where we often write t_w^* for $*(t, w)$), such that for arbitrary $s, t \in \mathbf{Tm}$ and any formula A ,

- (1) $s_w^* \cdot t_w^* \subseteq (s \cdot t)_w^*$;
- (2) $s_w^* \cup t_w^* \subseteq (s + t)_w^*$;
- (3) $(t, A) \in \mathbf{CS} \implies A \in t_w^*$;
- (4) $s : s_w^* \subseteq (!s)_w^*$.

Furthermore, it has to satisfy the following monotonicity conditions:

- (M1) $p_w^* = 1$ and $w \leq v \implies p_v^* = 1$;
- (M2) $w \leq v \implies t_w^* \subseteq p_v^*$.

Strictly speaking we should use the notion of a **CS** basic evaluation because of condition (3) depends on a given **CS**. However, the constant specification will always be clear from the context and we can safely omit it. The same also holds for modular models (to be introduced later).

Definition 3.2 (Truth under Basic Evaluation). Let $\mathfrak{M} = (W, \leq, *)$ be a basic evaluation. For $w \in W$, we define $(\mathfrak{M}, w) \models A$ by induction on the formula A as follows:

- $(\mathfrak{M}, w) \not\models \perp$;
- $(\mathfrak{M}, w) \models p$ iff $*(p, w) = 1$;
- $(\mathfrak{M}, w) \models A \wedge B$ iff $(\mathfrak{M}, w) \models A$ and $(\mathfrak{M}, w) \models B$;
- $(\mathfrak{M}, w) \models A \vee B$ iff $(\mathfrak{M}, w) \models A$ or $(\mathfrak{M}, w) \models B$;
- $(\mathfrak{M}, w) \models A \rightarrow B$ iff $(\mathfrak{M}, v) \models B$ for all $v \geq w$ with $(\mathfrak{M}, v) \models A$;
- $(\mathfrak{M}, w) \models t : A$ iff $A \in t_w^*$.

Lemma 3.3 (Monotonicity). For any basic evaluation $\mathfrak{M} = (W, \leq, *)$, states $w, v \in W$ and any formula A :

$$(\mathfrak{M}, w) \models A \text{ and } w \leq v \implies (\mathfrak{M}, v) \models A.$$

Definition 3.4 (Factive Evaluation). A basic evaluation $\mathfrak{M} = (W, \leq, *)$ is called *factive* iff

$$A \in t_w^* \implies (\mathfrak{M}, w) \models A$$

for all formulas A , all justification terms t and all states $w \in W$.

Definition 3.5 (Basic modular model). A *basic modular model* is a basic evaluation $(W, \leq, *)$ that is factive.

We say that a formula A is *valid with respect to basic modular models* (in symbols $\models_{\text{basicmodular}} A$) if for any basic modular model $\mathfrak{M} = (W, \leq, *)$ and any $w \in W$ we have $(\mathfrak{M}, w) \models A$.

Lemma 3.6 (Soundness of iJT4_{CS} with respect to basic modular models). For every formula A :

$$\vdash A \implies \models_{\text{basicmodular}} A$$

In order to show completeness, we need some auxiliary definitions and lemmas.

Definition 3.7. We call a set of formulas Δ *prime* iff it satisfies the following conditions:

- (i) Δ has the disjunction property, i.e., $A \vee B \in \Delta \implies A \in \Delta$ or $B \in \Delta$;
- (ii) Δ is deductively closed, i.e., for any formula A , if $\Delta \vdash A$, then $A \in \Delta$;
- (iii) Δ is consistent, i.e., $\perp \notin \Delta$.

From now on, we will use Σ, Δ, Γ for prime sets of formulas.

Lemma 3.8. Let N be an arbitrary set of formulas and let A, B and C be formulas. If

$$N \cup \{A \vee B\} \not\vdash C, \text{ then } N \cup \{A\} \not\vdash C \text{ or } N \cup \{B\} \not\vdash C.$$

Proof. By contraposition. Assume that

$$N \cup \{A\} \vdash C \text{ and } N \cup \{B\} \vdash C$$

Then there are finite subsets $N_1 \subseteq N \cup \{A\}$ and $N_2 \subseteq N \cup \{B\}$ such that

$$\vdash \bigwedge N_1 \rightarrow C \text{ and } \vdash \bigwedge N_2 \rightarrow C$$

Now let $N'_1 := N_1 \setminus \{A\}$ and $N'_2 := N_2 \setminus \{B\}$. Then N'_1, N'_2 are finite subsets of N , and

$$\vdash \bigwedge (N'_1 \cup \{A\}) \rightarrow C \text{ and } \vdash \bigwedge (N'_2 \cup \{B\}) \rightarrow C.$$

So

$$\vdash \bigwedge N'_1 \rightarrow (A \rightarrow C) \text{ and } \vdash \bigwedge N'_2 \rightarrow (B \rightarrow C).$$

Strengthening the antecedent, we get

$$\vdash \bigwedge (N'_1 \cup N'_2) \rightarrow (A \rightarrow C) \text{ and } \vdash \bigwedge (N'_1 \cup N'_2) \rightarrow (B \rightarrow C))$$

and, therefore,

$$\vdash \bigwedge (N'_1 \cup N'_2) \rightarrow ((A \rightarrow C) \wedge (B \rightarrow C)).$$

By propositional reasoning we get

$$\vdash \bigwedge (N'_1 \cup N'_2) \rightarrow ((A \vee B) \rightarrow C),$$

which means that

$$\vdash \bigwedge (N'_1 \cup N'_2 \cup \{A \vee B\}) \rightarrow C.$$

Since N'_1 and N'_2 are finite subsets of N , $(N'_1 \cup N'_2 \cup \{A \vee B\})$ is a finite subset of $N \cup \{A \vee B\}$, so by definition

$$N \cup \{A \vee B\} \vdash C. \quad \square$$

Theorem 3.9 (Prime Lemma). Let B be a formula and let N be a set of formulas such that $N \not\vdash B$. Then there exists a prime set Π with $N \subseteq \Pi$ and $\Pi \not\vdash B$.

Proof. Let $(A_n)_{n \in \mathbb{N}}$ be an enumeration of all formulas.

Now we define $N_0 := N$,

$$N_{i+1} := \begin{cases} N_i \cup \{A_i\}, & \text{if } N_i \cup \{A_i\} \not\vdash B \\ N_i, & \text{otherwise} \end{cases}$$

and finally

$$N^* := \bigcup_{i \in \mathbb{N}} N_i$$

By induction in i , one can easily show that for all $i \in \mathbb{N}$: $N_i \not\vdash B$ and, therefore, $N^* \not\vdash B$.

It remains to show that N^* is prime. We have the following:

- $\perp \notin N^*$: Since $\perp \notin N_i$ for all $i \in \mathbb{N}$, which can be shown by induction on i .
- N^* is deductively closed: Assume it is not, i.e., there is a formula A with

$$N^* \vdash A \text{ but } A \notin N^*$$

Since $N^* \vdash A$ but $N^* \not\vdash B$, we know that

$$N^* \cup \{A\} \not\vdash B$$

Otherwise, by the deduction theorem 2.4

$$N^* \vdash A \rightarrow B \text{ and } N^* \vdash A$$

so by propositional reasoning,

$$N^* \vdash B, \text{ which contradicts our observation above.}$$

Since $(A_n)_{n \in \mathbb{N}}$ is an enumeration of all formulas, there is some i such that $A = A_i$. But then

$$N_i \cup \{A_i\} \not\vdash B.$$

So by construction

$$N_{i+1} = N_i \cup \{A_i\}$$

and, therefore,

$$A = A_i \in N_{i+1} \subseteq N^*,$$

which contradicts our assumption.

- N^* has the disjunction property: Assume that $C \vee D \in N^*$. Then there is some i such that $C \vee D = A_i$ and there are i_1, i_2 such that

$$C = A_{i_1} \text{ and } D = A_{i_2}$$

Now we have

$$N^* = N^* \cup \{C \vee D\} \not\vdash B$$

By the lemma above it follows that

$$N^* \cup \{C\} \not\vdash B \text{ or } N^* \cup \{D\} \not\vdash B$$

In the first case, we have that

$$N_{i_1} \cup \{A_{i_1}\} \not\vdash B$$

so by the definition of N_{i_1+1} ,

$$N_{i_1+1} = N_{i_1} \cup \{A_{i_1}\} = N_{i_1} \cup \{C\}$$

which means that $C \in N_{i_1+1}$ and therefore $C \in N^*$. The second case is analogous. \square

Lemma 3.10. Let Δ be a prime set and t be a justification term. Then

$$t^{-1}\Delta \subseteq \Delta.$$

Proof. Let $A \in t^{-1}\Delta$. Then $t : A \in \Delta$. Since Δ is deductively closed, it contains all axioms, thus $t : A \rightarrow A \in \Delta$. Again, since Δ is deductively closed, it follows by (MP) that $A \in \Delta$. \square

Definition 3.11 (Canonical Basic Modular Model). The canonical basic modular model is

$$B^{can} := (W^{can}, \leq^{can}, *^{can})$$

where

- (i) $W^{can} := \{\Delta \subseteq \mathcal{L}_J : \Delta \text{ is prime}\}$
- (ii) $\leq^{can} := \subseteq$
- (iii) $*^{can}(p, \Delta) = 1$ iff $P \in \Delta$
- (iv) $*^{can}(t, \Delta) := t^{-1}\Delta := \{A \mid t : A \in \Delta\}$

Lemma 3.12. B^{can} is a basic evaluation.

Proof. $W \neq \emptyset$: By the consistency of iJT4_{CS} we have that $\emptyset \not\vdash \perp$, it follows by the prime lemma 3.9 that there exists a prime set, so $W^{can} \neq \emptyset$.

Next, we check the conditions on the sets of formulas t_w^{*can} .

- (1) $s_w^{*can} \cdot t_w^{*can} \subseteq (s \cdot t)_w^*$. Let $A \in s_w^{*can} \cdot t_w^{*can}$. Then there is a formula $B \in t_w^{*can}$ such that $B \rightarrow A \in s_w^{*can}$. So $s : B \rightarrow A \in w$ and $t : B \in w$. Since w is a prime set, it is deductively closed, so it contains the axiom $s : (B \rightarrow A) \rightarrow (t : B \rightarrow s \cdot t : A)$. Again since w is deductively closed, it follows by (MP) that $s \cdot t : A \in w$, so $A \in (s \cdot t)^{-1}w = (s \cdot t)_w^{*can}$.
- (2) $s_w^{*can} \cup t_w^{*can} \subseteq (s+t)_w^{*can}$. Let $A \in s_w^{*can} \cup t_w^{*can}$. Case 1: $A \in s_w^{*can} = s^{-1}w$. Then $s : A \in w$. Since w is deductively closed, it contains the axiom $s : A \rightarrow (s+t) : A$. Thus by (MP) we find $(s+t) : A \in w$, i.e., $A \in (s+t)^{-1}w = (s+t)_w^{*can}$. The second case is analogous.

- (3) $(t, A) \in CS \implies A \in t_w^{*can}$. By axiom necessitation we have that $\text{iJT4}_{CS} \vdash t : A$, so $w \vdash t : A$. Since w is deductively closed, it follows that $t : A \in w$, so $A \in t^{-1}w = t_w^{*can}$.
- (4) $s : s_w^{*can} \subseteq (!s)_w^{*can}$. Let $A \in s : s_w^{*can}$. Then A is of the form $s : B$ for some formula $B \in s_w^{*can} = s^{-1}w$, i.e., $s : B \in w$. We find that the axiom $(s : B) \rightarrow !s : (s : B) \in w$, so $!s : (s : B) \in w$, which means that $s : B \in (!s)^{-1}w = (!s)_w^{*can}$.

Now we check the monotonicity conditions.

(M1) Assume that $*(p, \Gamma) = 1$ and $\Gamma \subseteq \Delta$. By the definition of $*$ we have that $p \in \Gamma$, so $p \in \Delta$ hence $*(p, \Delta) = 1$.

(M2) Now assume that $\Gamma \subseteq \Delta$. Then $t^{-1}\Gamma \subseteq t^{-1}\Delta$ which means $t_\Gamma^* \subseteq t_\Delta^*$. \square

Lemma 3.13 (Truth Lemma). For any formula A and any prime set Δ :

$$A \in \Delta \iff (*^{can}, \Delta) \models A$$

Proof. By induction on the formula A . We distinguish the following cases.

1. $A = p$ or $A = \perp$. By definition.
2. $A = B \wedge C$. Assume that $B \wedge C \in \Delta$. Since Δ is deductively closed, we have $B \in \Delta$ and $C \in \Delta$, so it follows by the induction hypothesis that $(*^{can}, \Delta) \models B$ and $(*^{can}, \Delta) \models C$.

For the other direction assume that $(*^{can}, \Delta) \models B \wedge C$, so $(*^{can}, \Delta) \models B$ and $(*^{can}, \Delta) \models C$. By the induction hypothesis, we get that $B \in \Delta$ and $C \in \Delta$. Since Δ is deductively closed, it follows that $B \wedge C \in \Delta$.

3. $A = B \vee C$. Assume that $B \vee C \in \Delta$. Since Δ has the disjunction property, it follows that $B \in \Delta$ or $C \in \Delta$, so by the induction hypothesis, $(*^{can}, \Delta) \models B$ or $(*^{can}, \Delta) \models C$, so $(*^{can}, \Delta) \models B \vee C$.

For the other direction assume that $(*^{can}, \Delta) \models B \vee C$. Then

$$(*^{can}, \Delta) \models B \text{ or } (*^{can}, \Delta) \models C,$$

so by the induction hypothesis, $B \in \Delta$ or $C \in \Delta$. Since Δ is deductively closed, it follows that $B \vee C \in \Delta$.

4. $A = B \rightarrow C$. Assume that $B \rightarrow C \in \Delta$. We have to show that $(*^{can}, \Delta) \models B \rightarrow C$, so let Γ be a prime set such that $\Delta \subseteq \Gamma$ and $(*^{can}, \Gamma) \models B$. It follows by the induction hypothesis that $B \in \Gamma$, and

since $B \rightarrow C \in \Gamma$ and Γ is deductively closed, we have that $C \in \Gamma$. Applying the induction hypothesis again, we get that $(*^{can}, \Gamma) \models C$.

For the other direction assume that $(*^{can}, \Delta) \models B \rightarrow C$. We have to show that $B \rightarrow C \in \Delta$. Assume for a contradiction that $B \rightarrow C \notin \Delta$. Since Δ is deductively closed, it follows that $\Delta \not\models B \rightarrow C$. It follows by the deduction theorem 2.4 that $\Delta \cup \{B\} \not\models C$. By the prime lemma 3.9, there is a prime set Γ such that $\Delta \cup \{B\} \subseteq \Gamma$ and $\Gamma \not\models C$, so in particular, $C \notin \Gamma$. By the induction hypothesis it follows that $(*^{can}, \Gamma) \models B$ and $(*^{can}, \Gamma) \not\models C$, contradicting our assumption that $(*^{can}, \Delta) \models B \rightarrow C$.

5. $A = t : B$. We have

$$t : B \in \Delta \iff B \in t^{-1}\Delta = *^{can}(t, \Delta) \iff (*^{can}, \Delta) \models t : B.$$

□

Lemma 3.14. B^{can} is a basic modular model.

Proof. We only have to show factivity, for which we use the truth lemma. Assume that

$$A \in *^{can}(t, \Delta) = t^{-1}\Delta.$$

By Lemma 3.10 we know that $t^{-1}\Delta \subseteq \Delta$, so we have $A \in \Delta$. By the truth lemma for the canonical basic modular model, we can conclude that $(*^{can}, \Delta) \models A$. So factivity is shown. □

Theorem 3.15 (Completeness of iJT4_{CS} with respect to basic modular models). For any formula A :

$$\models_{\text{basicmodular}} A \implies \text{iJT4}_{CS} \vdash A$$

Proof. By contraposition. Assume that $\text{iJT4}_{CS} \not\vdash A$. By the prime lemma 3.9, there exists a prime set Δ such that $\Delta \not\models A$. In particular, $A \notin \Delta$. By the truth lemma 3.13, it follows that

$$(*^{can}, \Delta) \not\models A$$

since this structure is a basic modular model, it follows that

$$\not\models_{\text{basicmodular}} A.$$

□

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